CS477
Logic

First order logics

First order theories

Single models vs Axioms

Godel’s strong completeness theorem

Quantifier free fragment

Decidable theories

First-order logic

$Var = \{x_0, x_1, \ldots\}$

$$\Sigma = (\mathbb{C}, \mathbb{F}, \mathbb{R})$$

$\{\leq, =, <, |, \text{int?}, \text{prime}\}$

Arithmetic

$\{0, 1, \ldots\}$ π

$\{+ : 2,\ * : 2,\ - : 2,\ -' : 1\}$

$\alpha \Rightarrow \beta \equiv \neg\alpha \vee \beta$

$c \in \mathbb{C}$

FO: Terms $t_1 \ldots t_n$: $c \mid f(t_1, \ldots t_n) \mid x$

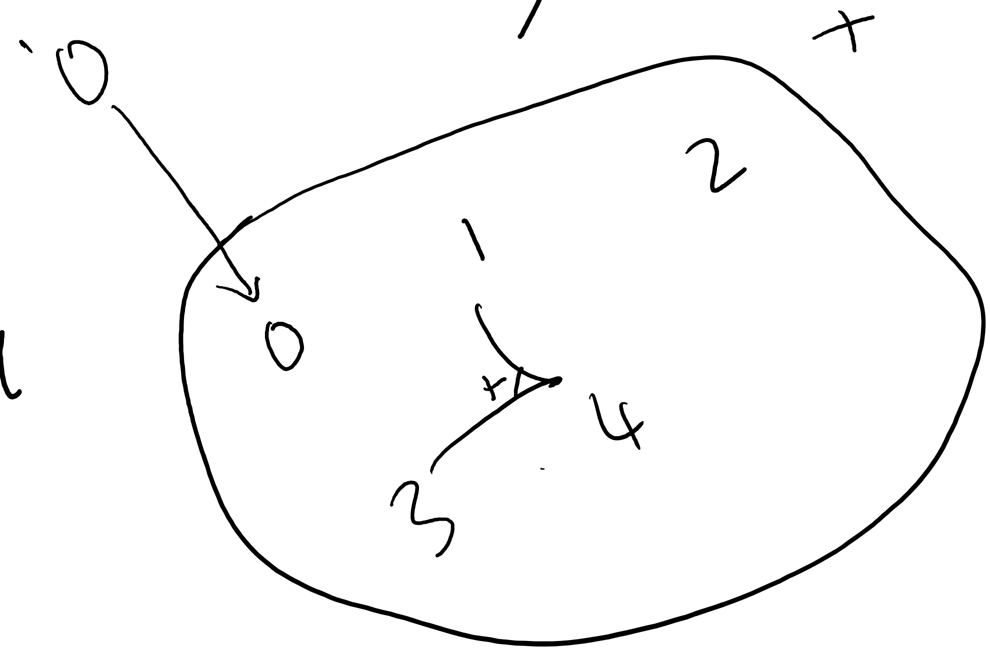
Formulas α, β : $t_1 = t_2 \mid R(t_1, \ldots t_n) \mid \alpha \vee \beta \mid \alpha \wedge \beta \mid \neg\alpha \mid$

$\exists x . \alpha \mid \forall x . \alpha$

Models

$$(\mathcal{U}, [\![c]\!], [\![f]\!], [\![R]\!])$$

nonempty set



$$[\![f]\!] : \mathcal{U}^n \to \mathcal{U}$$

$$[\![R]\!] \subseteq \mathcal{U}^n$$

$$[\![c]\!] \in \mathcal{U}$$

$$M \models \varphi$$

φ is a sentence (formula with no free var)

$M, I \models \varphi$ $\qquad$ φ : formula

$I : \text{Vars} \to \mathcal{U}$

Validity : φ

Is φ true in every model?
(model and interpretation)

$\forall x \; x = x$

$\forall x y \quad R(x,y) \Leftrightarrow R(x,y)$

$\forall x \quad (R(x) \wedge S(x)) \Rightarrow R(x)$

Satisfiability φ

Does φ hold in some models?
(and interpretation)

φ is valid iff $\neg\varphi$ is not satisfiable

Dec./ computability

$\text{Valid}_\Sigma = \{$ FO formulas over Σ that are valid $\}$

$\text{Sat}_\Sigma = \{$ FO formulas over Σ that are satisfiable $\}$

Gödel's completeness theorem $\Rightarrow$ Valid_Σ is in r.e.

Valid_Σ is not decidable.

Sat_Σ is not r.e.

How do I reason in particular models?

Fix M.

$Th(M) = \underline{\underline{Valid_{\Sigma, M}}} := \{$ FO formulas that are true in M over our Σ $\}$

Theories using Axioms

$\mathcal{A}$ - sentences.
recursive set

$Th(\mathcal{A}) = \{ \varphi \mid \varphi$ is a FO-sentence and every model satisfying $\mathcal{A}$ also satisfies $\varphi \}$.

Axioms can never pin down one (infinite) model.

But axioms can pin down theories!

Complete theory:

$Th(\mathcal{A})$ is complete or $\mathcal{A}$ is complete if for every sentence φ either $\varphi \in Th(\mathcal{A})$ or $\neg\varphi \in Th(\mathcal{A})$

Arithmetic $(\{0,1\}, \{+\}, \{=, \leq\})$

Presburger arithmetic : Axioms that pin down the theory of Arithmetic (+)

$\circ M$

$Th(M) = \{\varphi \mid \varphi \text{ holds in } M\}$

$Th(M)$ is always complete.

<u>Consistent</u>: There is no φ such that $\varphi, \neg\varphi \in Th(M)$

Entailment closed: If $Th(M) \models \alpha$ then $\alpha \in Th(M)$

$Th(\mathcal{M}) = \{\varphi \mid \varphi \text{ holds in all models in } \mathcal{M}\}$

$Th(\mathcal{M})$ is not, in general, complete.

$Th(\mathcal{M})$ is consistent. ($\mathcal{M}$ is not empty)

$Th(\mathcal{M})$ is entailment closed.

$Th(\mathcal{M})$ is

$\mathcal{A}$ PresAr : $(\{0,1\}, \{+\}, \{=, \leq\})$

$\mathcal{A}_{PresAr}$

$Th(\mathcal{A}_{PresAr}) = Th($ Std. model of arithm $)$

Arithmetic $(\{0,1\}, \{+, *\}, \{=, \leq\})$

Gödel's Incompleteness Thm :

"There is no recursive set of axioms that capture "arithmetic"

~~No set of~~ If a set axioms captures all true sentences in arithmetic, then it is inconsistent.

$\mathcal{A}$ - axioms. $\models \varphi$ iff $\vdash_{PS} \varphi$ (Gödel's completeness thm)

$\mathcal{A} \models \varphi$ iff $\mathcal{A} \vdash_{PS} \varphi$ (Gödel strong completeness thm.)

Every theorem (modulo axioms $\mathcal{A}$) has a proof.

A — axioms (recursive)

A is complete.

If $A \models \varphi$ then φ has a proof.

$Th(A)$ is r.e.

If A is complete (and consistent), $Th(A)$ is decidable.

Peano Arithmetic: $(\{0, 1\}, \{+, *\}, \{=, \leq\})$

Peano Axioms

Peano Arithmetic is undecidable (as well).

Quantifier free fragment is undecidable.

$\forall x, y. \ (xy > 10 \wedge \ldots)$

$A = \emptyset$

$Th_\Sigma(\emptyset) = \{ \varphi \mid \models \varphi$ i.e. φ is valid$\}$

is decidable?

No.

But Quant free fragment is decidable.